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Group Theoretic and Similarity Analysis of Hyperbolic Partial Differential Equations

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Multiparameter dimensional groups of transformations are applied to a system of quasilinear hyperbolic partial differential equations along with their auxiliary conditions. It is shown that for such a class of problems the similarity characteristic relationship can be stated and the location of the wavefront in terms of similarity variable can be determined. Two theorems are stated and their proofs given. Furthermore, it is shown that similarity transformation is one-to-one and onto. Similarity analysis of two problems, arising in wave propagation in nonlinear-non-homogeneous rods, is presented in the light of theory presented in the paper.

1. INTRODUCTION AND BACKGROUND THEORY

With the aim of constructing a theory of integrating ordinary differential equations, Sophus Lie investigated continuous groups of transformations systematically in the later part of the ninteenth century [1]. Since its inception Lie's work has had profound affect in its implications both on mathematical and physical sciences. Based on the works of Sophus Lie, Ovsiannikov [2, 3] made use of infinitesimal group of transformations to construct continuous groups under which a given partial differential equation is invariant. These groups, in general, lead to the determination of similarity transformations of the equations under consideration.

A simpler approach, more convenient from the point of view of solution of partial boundary value problems in Engineering, is provided by the use of dimensional groups of transformations. Birkhoff [4] was one of the first to apply these groups to the similarity solution of problems in Hydrodynamics. Birkhoff's work was further extended for application to partial differential equations by Morgan and Michal [5, 6]. Moran and Gaggioli [7] and later Moran and Marshek [8] extended the analysis and

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results to boundary and initial value problems in Fluid Mechanics. Seshadri and Singh [9] made use of the similarity characteristic relationship at the wavefront in the case of wave propagation in a nonlinear elastic rod to reduce the system to a two point boundary value problem.

In this paper quasilinear hyperbolic partial differential equations of second order are considered to show that conformal invariance of such equations under a multiparameter-dimensional group of transformations implies the conformal invariance of their characteristics. This analysis leads to similarity characteristic relationship which is used in turn to formulate a boundary condition at the wavefront. Furthermore, it is shown that similarity transformation is one-to-one and onto. Some examples for application to the problem of unidirectional wavemotion are given at the end.

In the theoretical background the central feature of dimensional-similarity analysis is the use of a continuous r-parameter group of transformations [10]

$$\bar{Z}_i = f_i(Z_1, Z_2, \ldots, Z_n; A_1, A_2, \ldots, A_r), \; (i = 1, \ldots, n) \tag{1.1a}$$

and the absolute invariants π's which satisfy,

$$\pi(Z_1, Z_2, \ldots, Z_n) = \pi(\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n). \tag{1.1b}$$

The A's of (1.1a) are a set of parameters of which more is said later. The variables Z_i of (1.1a) correspond to the variables appearing in the set of governing equations under consideration, i.e., correspond to the variables of a set of differential equations and its associated boundary and/or initial conditions.

The first step in the application of the group-theoretic approach to dimensional-similarity analysis is the establishment of a group under the transformations of which the set of governing equations is invariant in form. Next, the absolute invariants of the group are used to express the governing equations in terms of a less number of variables.

Experience reveals that for a wide range of problems in engineering and sciences a sufficiently general group of transformations is provided by a subclass of r-parameter groups (1.1a) with the form [11]

$$\bar{Z}_i = A_1^{\gamma_{i1}} \cdots A_r^{\gamma_{ir}} Z_i \qquad (i = 1, \ldots, n). \tag{1.2}$$

Many of the manipulative difficulties inherent in the foregoing approach are eliminated by initiating an analysis with a group of transformations with the form of (1.2). With (1.2) the initial objective is merely to establish restrictions imposed upon the exponents $\gamma_{i\alpha}$ ($\alpha = 1,\ldots, r$) by a given set of governing equations in order that the set be invariant in form.

In the engineering applications each of the variables in any set of governing equations under consideration is regarded as being in one of three distinct categories: (i) dependent, (ii) independent, (iii) physical. For instance, the position and time coordinates, x, y, z, t may be identified as independent variables; components of displacement vector and stress tensor may be identified as dependent variables, and finally the density ρ, material constants E, ν may be identified as physical variables or parameters.

In recognition of the foregoing three categories for variables, the class of r-parameter groups (1.2) can be written somewhat more explicitly. Thus consider r-parameter groups with the form

$$G_r^{E_k}: \left\{ \begin{array}{l} G_r: \left\{ \begin{array}{ll} \bar{Z}_j = A_1^{a_{j1}} \cdots A_r^{a_{jr}} Z_j & (j = 1,\ldots, n \geqslant 1), \quad (1.3) \\ S_r: \left\{ \begin{array}{ll} \bar{X}_k = A_1^{b_{k1}} \cdots A_r^{b_{kr}} X_k & (k = 1,\ldots, m \geqslant 1), \quad (1.4) \\ \bar{Y}_l = A_1^{c_{l1}} \cdots A_r^{c_{lr}} Y_l & (l = 1,\ldots, p \geqslant 0), \quad (1.5) \end{array} \right. \end{array} \right. \\ \left[\dfrac{\partial^\lambda \bar{Z}_j}{\partial(\bar{X}_1)^{\lambda_1} \cdots \partial(\bar{X}_m)^{\lambda_m}} \right] = [A_1^{\beta_{j1}} \cdots A_r^{\beta_{jr}}] \left[\dfrac{\partial^\lambda Z_j}{(\bar{X}_1)^{\lambda_1} \cdots (\bar{X}_m)^{\lambda_m}} \right], \quad (1.6) \\ \text{where } 1 \leqslant \lambda = \sum_{k=1}^{m} \lambda_k \leqslant \kappa, \quad (1.7) \\ \text{and } \beta_{j\alpha} \equiv a_{j\alpha} - \sum_k \lambda_k b_{k\alpha} \quad (\alpha = 1,\ldots, r). \quad (1.8) \end{array} \right.$$

In the above Z's are associated with the dependent variables of a set of governing equations and are differentiable functions of the X_k up to any required order, the X's are associated with the independent variables, and Y's are associated with the physical parameters, $1 \leqslant r \leqslant m + p$, and each of the group parameters A_α ($\alpha = 1,\ldots, r$) is a positive real variable and S_r form an r-parameter subgroup of G_r. If the transformations of the partial derivatives of the Z_j with respect to the X_i are appended to those of (1.3), (1.4), (1.5), it can be shown that the resulting set of transformations also forms a continuous r-parameter group. The new groups constructed in this manner are called enlargement of the group G_r and are denoted by $G_r^{E_1},\ldots, G_r^{E_k}$ according as the transformations of the first, second,..., kth order of partial derivatives of the Z_j are added successively to those of G_r. Further, it is to be understood that the parameters A_α are to be essential [10]. Thus, to give an example, if in the transformations of some group the parameters A_1 and A_2 would always occur as the product $[A_1 A_2]$, they would then be replaced by a single essential parameter, say $\hat{A}$. For S_r, and hence G_r, to involve r essential parameters it is both necesary and sufficient for the rank of the $([m+p]\times r)$ matrix BC, obtained by augmenting $(m\times r)$ matrix $B=[b_{k\alpha}]$ with $(p\times r)$ matrix $C=[c_{l\alpha}]$, to be r [11], where

$$BC=\begin{bmatrix} b_{k1},..., b_{kr} \\ c_{l1},..., c_{lr} \end{bmatrix}, \qquad k=1, 2,..., m,\ l=1, 2,..., p.$$

The matrix C is assumed to have rank s, $s\leqslant r$. The assumption about r, $1\leqslant r\leqslant m+p$, given above is clear now because of essential parameters [8, 11].

The earlier discussion [11] has suggested the importance of the role of invariant solution of a generalized dimensional analysis. One of the principal results of [11] is summarized in Theorem 1, which is formulated here in terms of an r-parameter group (1.3)–(1.8).

THEOREM 1. *If the function I_j is absolutely invariant in form under an r-parameter group of transformations* (1.3)–(1.5); *i.e., if $Z_j=I_j\ (X_1,..., X_m;\ Y_1,..., Y_p)$ transforms to $\bar{Z}_j=I_j\ (\bar{X}_1,..., \bar{X}_m;\ \bar{Y}_1,..., \bar{Y}_p)$, then*

(i) *$Z_j=I_j(\cdots)$ is equivalent to a relationship in fewer variables,*

$$\begin{aligned}&\Pi_j(Z_j, X_1,..., X_m;\ Y_1,..., Y_p)\\ &\quad=F_j(\pi_1(X_1,..., X_m;\ Y_1,..., Y_p),..., \pi_\delta(X_1,..., X_m;\ Y_1,..., Y_p)),\end{aligned} \tag{1.9}$$

wherein

$$\delta=[m+p-r]>0 \qquad \textit{and} \qquad \{\Pi_j, \pi_1,..., \pi_\delta\}$$

are independent absolute invariants of (1.3)–(1.5).

(ii) *If $\delta=0$ in* (1.3)–(1.5), *there exist constants $\{K_j\}$ such that*

$$\Pi_j=K_j. \tag{1.10}$$

Theorem 1 plays the role of the well known Pi-Theorem of conventional dimensional analysis [12].

In Theorem 1 the $[m+p-r]$ independent absolute invariants π_δ can be given by expressions of the form

$$\pi_\rho=[X_1]^{B_{\rho 1}}\cdots[X_m]^{B_{\rho m}}[Y_1]^{C_{\rho 1}}\cdots[Y_p]^{C_{\rho p}}, \tag{1.11}$$

where $\rho=1,..., [m+p-r]$ and the sets $\{B_{\rho k};C_{\rho l}\}$ provide linearly independent solutions for

$$\sum_{k=1}^{m} B_{\rho k}b_{k\alpha}+\sum_{l=1}^{p} C_{\rho l}C_{l\alpha}=0, \tag{1.12}$$

while the Π_j $(j=1,\ldots,n)$ have the form

$$\Pi_j=\{[Z_1]^{A_{j1}}\cdots[Z_n]^{A_{jn}}[X_1]^{B_{j1}}\cdots[X_m]^{B_{jm}}[Y_1]^{C_{j1}}\cdots[Y_P]^{C_{jp}}\}, \tag{1.13}$$

where the sets $\{A_{j\alpha}; B_{jk}; C_{jl}\}$ provide linearly independent solutions to

$$\sum_{\gamma=1}^{n} A_{j\gamma}a_{\gamma\alpha}+\sum_{k=1}^{m} B_{jk}b_{k\alpha}+\sum_{l=1}^{p} C_{jl}c_{l\alpha}=0. \tag{1.14}$$

It is noted that the pi's in (1.11) and (1.13) are absolute invariants of the group G_r, that is, functions such that under the transformations (1.3)–(1.5),

$$\begin{aligned}&\Pi(\bar{X}_1,\ldots,\bar{X}_m;\bar{Y}_1,\ldots,\bar{Y}_p;\bar{Z}_1,\ldots,\bar{Z}_n)\\&\quad=\Pi(X_1,\ldots,X_m;Y_1,\ldots,Y_p;Z_1,\ldots,Z_n).\end{aligned} \tag{1.15}$$

For the proof of the Theorem 1 see [5, 8, 11]. The Π's are also termed as dimensionless products.

For applications very important is the case when the rank r of the matrix BC associated with an r-parameter group (1.3)–(1.5) is greater than the rank s of the matrix C, i.e., $r>s$. In this case we have the following theorem [8]:

THEOREM 2. *If and only if $r>s$, the set of $[n+m+p-r]$-independent absolute invariants required by Theorem* 1 *may be obtained in the form,*

$$\Pi_j=Z_j[X_\varepsilon]^{A_{j\varepsilon}}\cdots[X_m]^{A_{jm}}[Y_1]^{\lambda_{j1}}\cdots[Y_s]^{\lambda_{js}},\qquad (j=1,\ldots,n), \tag{1.16}$$

$$\hat{\pi}_\sigma=X_\sigma[X_\varepsilon]^{\Gamma_{\sigma\varepsilon}}\cdots[X_m]^{\Gamma_{\sigma m}}[Y_1]^{\gamma_{\sigma 1}}\cdots[Y_s]^{\gamma_{\sigma s}},\qquad (\sigma=1,\ldots,[m+s-r]), \tag{1.17}$$

$$\tilde{\pi}_\rho=Y_\rho[Y_1]^{\delta_{\rho 1}}\cdots[Y_s]^{\delta_{\rho s}},\qquad (\rho=[s+1],\ldots,p), \tag{1.18}$$

wherein $\varepsilon\equiv[m+s-r+1]\leqslant m$ and the sets $\{A_{j\alpha},\lambda_{j\omega};\ \Gamma_{\sigma\alpha},\gamma_{\sigma\omega};\ \delta_{\rho\omega}\}$ provide linearly independent solutions to

$$\sum_{\alpha=\varepsilon}^{m} A_{j\alpha}\begin{bmatrix}b_{\alpha 1}\\ b_{\alpha 2}\\ \vdots\\ b_{\alpha r}\end{bmatrix}+\sum_{\omega=1}^{s}\lambda_{j\omega}\begin{bmatrix}c_{\omega 1}\\ c_{\omega 2}\\ \vdots\\ c_{\omega r}\end{bmatrix}=-\begin{bmatrix}a_{j1}\\ a_{j2}\\ \vdots\\ a_{jr}\end{bmatrix}\qquad (j=1,\ldots,n), \tag{1.19}$$

$$\sum_{\alpha=\varepsilon}^{m} \Gamma_{\sigma\alpha}\begin{bmatrix}b_{\alpha 1}\\ b_{\alpha 2}\\ \vdots\\ b_{\alpha r}\end{bmatrix}+\sum_{\omega=1}^{s}\gamma_{\sigma\omega}\begin{bmatrix}c_{\omega 1}\\ c_{\omega 2}\\ \vdots\\ c_{\omega r}\end{bmatrix}=-\begin{bmatrix}b_{\sigma 1}\\ b_{\sigma 2}\\ \vdots\\ b_{\sigma r}\end{bmatrix}\qquad (\sigma=1,\ldots,[m+s-r]), \tag{1.20}$$

$$\sum_{\omega=1}^{s} \delta_{\rho\omega} \begin{bmatrix} c_{\omega 1} \\ c_{\omega 2} \\ \vdots \\ c_{\omega r} \end{bmatrix} = - \begin{bmatrix} c_{\rho 1} \\ c_{\rho 2} \\ \vdots \\ c_{\rho r} \end{bmatrix} \qquad (\rho = [s+1], \dots, p). \tag{1.21}$$

From the Theorem 2 we see that for special case $p = s$ there is no absolute invariant determined solely from physical variables.

We can also observe that when Theorem 1 is applied via (1.3)–(1.5) for which $r = s$, the outcome of the application can only lead to a reduction in the number of physical variables, and cannot lead to a reduction in the number of independent variables. In the case when $r > s$ it is clear from Theorem 2 that the number of independent variables $\hat{\pi}$ in F_j is fewer than the number of independent variables in the original relationship I_j. For the proof of theorem 2 see [8].

DEFINITION 1. $\hat{\pi}_\sigma$ is termed a similarity variables whenever at least one of the exponents $\Gamma_{\sigma\alpha}$ ($\alpha = \varepsilon, \dots, m$) is nonzero; and is termed a normalized variable whenever each of the exponents $\Gamma_{\sigma\alpha}$ is zero.

In the further discussion only the case when $r > s$ is to be considered, i.e., when the similarity transformation for (1.3)–(1.5) under G_r can be obtained.

2. QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER 2 WITH TWO INDEPENDENT VARIABLES

Definition 2 [5]. By a differential form of the kth-order in m independent variables is meant a function, usually in class $C^{(1)}$ or greater, of the form

$$\psi\left(x_1, \dots, x_m, y_1, \dots, y_n, \dots, \frac{\partial^k y_1}{\partial (x_1)^k}, \dots, \frac{\partial^k y_n}{\partial (x_m)^k}\right), \tag{2.1}$$

whose arguments are the variables $x_1, \dots, x_m$, functions $y_1, \dots, y_n$ dependent on them, and the partial derivatives of y_j with respect to x_i up to the kth order, $j = 1, 2, \dots, n$, $i = 1, 2, \dots, m$.

DEFINITION 3 [7]. Consider a differential form ψ, whose arguments $\xi_1, \dots, \xi_p$ are the variables $x_1, \dots, x_m$, functions $y_1, \dots, y_n$, and the derivatives thereof up to the kth order. ψ is said to be conformally invariant under $G_r^{\xi_k}$ if

$$\psi(\bar{\xi}_1, \dots, \bar{\xi}_p) = F(\xi_1, \dots, \xi_p; A_1, \dots, A_r)\, \psi(\xi_1, \dots, \xi_p), \tag{2.2}$$

where ψ is exactly the same function of the ξ's as it is of the $\bar{\xi}$'s and F is some function of the ξ's and the parameters A's.

A partial differential equation of order r is quasilinear if and only if it is linear in the rth-order derivatives of the unknown function ϕ. Thus a quasilinear second-order equation with two independent variables x, t has the form

$$a_{11}\frac{\partial^2\phi}{\partial x^2}+2a_{12}\frac{\partial^2\phi}{\partial x\,\partial t}+a_{22}\frac{\partial^2\phi}{\partial t^2}+B=0, \tag{2.3}$$

where a_{11}, a_{12}, a_{22}, and B are suitably differentiable real functions of x, t, ϕ, $\partial\phi/\partial x$, and $\partial\phi/\partial t$. With each quasilinear second-order equation (2.3) are associated the characteristic curves $x=x(\tau)$, $t=t(\tau)$, $z=z(\tau)$ on the solution surface $z=\phi(x,t)$ such that $t=t(x)$ satisfies the following equation

$$a_{11}\left(\frac{dt}{dx}\right)^2-2a_{12}\frac{dt}{dx}+a_{22}=0. \tag{2.4}$$

Now let us take the class of r-parameter groups with the form (1.3)–(1.8) for $n=1$, $m=2$. Thus the following theorem holds:

THEOREM 3. *If the quasilinear partial differential equation of second order*, (2.3), *is conformally invariant under r-parameter groups of transformations* (1.3)–(1.8) *then the equation of characteristics* (2.4) *is conformally invariant under the same groups of transformations, too.*

Proof. The r-parameter group G_r (1.3)–(1.5) for our case has the following form:

$$G_r\colon\left\{\begin{array}{l} \bar\phi=A_1^{a_{11}}\cdots A_r^{a_{1r}}\phi,\\ S_r\colon\left\{\begin{array}{ll}\bar X=A_1^{b_{11}}\cdots A_r^{b_{1r}}X, & \bar t=A_1^{b_{21}}\cdots A_r^{b_{2r}}t,\\ \bar Y_l=A_1^{c_{l1}}\cdots A_r^{c_{lr}}Y_l & (l=1,\dots,p\geqslant 0),\end{array}\right.\end{array}\right. \tag{2.5}$$

while the enlargements of the group G_r are given below:

$$\frac{\partial\bar\phi}{\partial\bar x}=A_1^{(a_{11}-b_{11})}\cdots A_r^{(a_{1r}-b_{1r})}\frac{\partial\phi}{\partial x}, \tag{2.6}$$

$$\frac{\partial\bar\phi}{\partial\bar t}=A_1^{(a_{11}-b_{21})}\cdots A_r^{(a_{1r}-b_{2r})}\frac{\partial\phi}{\partial t}, \tag{2.7}$$

$$\frac{\partial^2\bar\phi}{\partial\bar x^2}=A_1^{(a_{11}-2b_{11})}\cdots A_r^{(a_{1r}-2b_{1r})}\frac{\partial^2\phi}{\partial x^2}, \tag{2.8}$$

$$\frac{\partial^2\bar\phi}{\partial\bar t^2}=A_1^{(a_{11}-2b_{11})}\cdots A_r^{(a_{1r}-2b_{2r})}\frac{\partial^2\phi}{\partial t^2}, \tag{2.9}$$

$$\frac{\partial^2\bar\phi}{\partial\bar x\,\partial\bar t}=A_1^{(a_{11}-b_{11}-b_{21})}\cdots A_r^{(a_{1r}-b_{1r}-b_{2r})}\frac{\partial^2\phi}{\partial x\,\partial t}. \tag{2.10}$$

Transformations (2.5) to (2.10) constitute the group $G_r^{E_2}$.

Making the use of relations (2.5)–(2.7) the differentiable real functions $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{22}$, and $\bar{B}$ can be expressed as follows:

$$\bar{a}_{11}\left(\bar{x}, \bar{t}\,\bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right)$$

$$= \bar{a}_{11}\left(A_1^{b_{11}}\cdots A_r^{b_{1r}}x,\ A_1^{b_{21}}\cdots A_r^{b_{2r}}t,\ A_1^{a_{11}}\cdots A_r^{a_{1r}}\phi,\right.$$

$$\left.A_1^{(a_{11}-b_{11})}\cdots A_r^{(a_{1r}-b_{1r})}\frac{\partial\phi}{\partial x},\ A_1^{(a_{11}-b_{21})}\cdots A_r^{(a_{1r}-b_{2r})}\frac{\partial\phi}{\partial t}\right), \quad (2.11)$$

and in the same manner

$$\bar{a}_{12}\left(\bar{x}, \bar{t}\,\bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = \bar{a}_{12}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, A_2, \ldots, A_r\right), \quad (2.12)$$

$$\bar{a}_{22}\left(\bar{x}, \bar{t}\,\bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = \bar{a}_{22}\left(x, t\,\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right), \quad (2.13)$$

$$\bar{a}_{12}\left(\bar{x}, \bar{t}\,\bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = \bar{a}_{12}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right), \quad (2.14)$$

$$\bar{B}\left(\bar{x}, \bar{t}\,\bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = \bar{B}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right). \quad (2.15)$$

It turns out in the following that whenever the differential form (2.3) is invariant under the enlargement group $G_r^{E_2}$ (2.5)–(2.10) then the differential forms a_{ij} $(i, j = 1, 2)$ and B are invariant under $G_r^{E_2}$, too.

Making use of (2.6)–(2.15) the differential form (2.3), written in the "bar" variables, can be transformed as follows:

$$\bar{a}_{11}\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + 2\bar{a}_{12}\frac{\partial^2\bar{\phi}}{\partial\bar{x}\,\partial\bar{t}} + \bar{a}_{22}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} + \bar{B}$$

$$= \bar{a}_{11}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right) A_1^{(a_{11}-2b_{11})}\cdots A_r^{(a_{1r}-2b_{1r})}$$

$$\times\frac{\partial^2\phi}{\partial x^2} + 2\bar{a}_{12}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right)$$

$$A_1^{(a_{11}-b_{11}-b_{21})}\cdots A_r^{(a_{1r}-b_{1r}-b_{2r})}\frac{\partial^2\phi}{\partial x\,\partial t}$$

$$+ \bar{a}_{22}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1, \ldots, A_r\right) A_1^{(a_{11}-2b_{21})}\cdots A_r^{(a_{1r}-2b_{2r})}$$

$$\times\frac{\partial^2\phi}{\partial t^2} + \bar{B}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}\right). \quad (2.16)$$

The representation (2.16) is conformally invariant under the enlargement group $G_r^{E_2}$ (2.5)–(2.10) if the following relations hold:

$$\begin{aligned} &A_1^{(a_{11}-2b_{11})}A_2^{(a_{12}-2b_{12})}\cdots A_r^{(a_{1r}-2b_{1r})} \\ &\quad = A_1^{(a_{11}-b_{11}-b_{21})}A_2^{(a_{12}-b_{12}-b_{22})}\cdots A_r^{(a_{1r}-b_{1r}-b_{2r})} \\ &\quad = A_1^{(a_{11}-2b_{21})}A_2^{(a_{12}-2b_{22})}\cdots A_r^{(a_{1r}-2b_{2r})}, \end{aligned} \tag{2.17}$$

and simultaneously there has to exist a function F such that

$$\begin{aligned} \bar{a}_{11}\left(\bar{x}, \bar{t}\, \bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) &= F\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}; A_1,\dots, A_r\right) \\ &\quad \times a_{11}\left(x, t, \phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}\right), \end{aligned} \tag{2.18}$$

$$\bar{a}_{12}\left(\bar{x}, \bar{t}\, \bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = F(\cdots)\, a_{12}\left(x, t, \phi, \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial t}\right), \tag{2.19}$$

$$\bar{a}_{22}\left(\bar{x}, \bar{t}\, \bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}}, \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = F(\cdots)\, a_{22}\left(x, t, \phi, \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial t}\right), \tag{2.20}$$

$$\bar{B}\left(\bar{x}, \bar{t}\, \bar{\phi}, \frac{\partial\bar{\phi}}{\partial\bar{x}} \frac{\partial\bar{\phi}}{\partial\bar{t}}\right) = F(\cdots)\, B\left(x, t, \phi, \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial t}\right). \tag{2.21}$$

From the definition 3 and relations (2.18)–(2.21) we see that the differential forms a_{ij} $(i, j = 1, 2)$ and B are conformally invariant under $G_r^{E_2}$. The relations (2.17) are satisfied if and only if

$$b_{2\alpha} = b_{1\alpha} \qquad (\alpha = 1,\dots, r) \tag{2.22}$$

and for any

$$a_{1\alpha} \qquad (\alpha = 1,\dots, r).$$

Coming to the differential form (2.4) and making use of (2.18)–(2.22), we obtain the following relation

$$\begin{aligned} &\bar{a}_{11}\left(\frac{d\bar{t}}{d\bar{x}}\right)^2 - 2\bar{a}_{12}\frac{d\bar{t}}{d\bar{x}} + \bar{a}_{22} \\ &\quad = F(\cdots)\, a_{11} A_1^{2(b_{21}-b_{11})} A_2^{2(b_{22}-b_{12})}\cdots A_r^{2(b_{2r}-b_{1r})}\left(\frac{dt}{dx}\right)^2 \\ &\qquad - 2F(\cdots)\, a_{12} A_1^{(b_{21}-b_{11})}\cdots A_2^{(b_{22}-b_{1r})}\frac{dt}{dx} \\ &\qquad + F(\cdots)\, a_{22} = F(\cdots)\left[a_{11}\left(\frac{dt}{dx}\right)^2 - 2a_{12}\frac{dt}{dx} + a_{22}\right], \end{aligned} \tag{2.23}$$

where $F(\cdots)$ is the same as given in (2.18). From Definition 3 it follows that the differential form (2.4) is conformally invariant under $G_r^{E_2}$. This proves Theorem 3.

3. Quasilinear Hyperbolic Partial Differential Equations of Order 2 with Two Independent Variables

Let us consider the quasilinear hyperbolic partial differential equation of order 2 with two independent variables

$$a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\,\partial t}+a_{22}\frac{\partial^2 u}{\partial t^2}+B=0, \qquad x\geqslant 0,\ t\geqslant 0, \tag{3.1}$$

where a_{ij} $(i, j=1, 2)$ and B are initially differentiable real functions of x, t, u, $\partial u/\partial x$, and $\partial u/\partial t$, subjected to a boundary condition

$$\frac{\partial u}{\partial t}(x=0, t)=Lu_0(t), \qquad t>0, \tag{3.2}$$

or

$$u(x=0, t)=Lu_0(t), \qquad t>0, \tag{3.2a}$$

with the condition at the wave front as

$$u(x=x_w(t), t)=0, \qquad t>0, \tag{3.3}$$

where $x=x_w(t)$ defines the wavefront, and

$$u(x, t)=0 \quad \text{for} \quad x>x_w,\ t>0, \tag{3.4}$$

and initial conditions

$$u(x, t=0)=0, \qquad x>0 \tag{3.5}$$

$$\frac{\partial u}{\partial t}(x, t=0)=0, \qquad x>0. \tag{3.6}$$

The hyperbolic partial differential equation (3.1) has two distinct families of real characteristic described by the equation

$$\frac{dt}{dx}=(a_{12}\pm\sqrt{a_{12}^2-a_{11}a_{12}})/a_{11}, \tag{3.7}$$

where $a_{11}a_{22}-a_{12}^2<0$. Since waves propagate with a finite velocity c, it follows that any point more distant than characteristic crossing the point

(0, 0) from the boundary given by (3.7), is not affected by the presence of boundary condition and we therefore need not consider this region any further. Hence, the condition (3.4) will be satisfied identically and we shall chiefly treat the region in which the boundary condition (3.2) takes effect.

Let us assume that the differential form given by (3.1), boundary condition (3.2) and initial conditions (3.5) and (3.6) are conformally invariant under an r-parameter groups of transformations G_r, (2.5), and its enlargements (2.6)–(2.10). We know that the rank of the matrix BC is $r \leqslant 2+p$. Theorem 3 indicates that auxiliary condition (3.3) does not give any further restriction on the parameters $A_1,\ldots, A_r$. Thus, we note then that for such formulation of the problem the similarity transformation can be obtained. Theorems (1) and (2) indicate the manner in which it can be done and the conditions which must be satisfied by the exponents. Following Theorem 2 let us express the similarity transformation in the general form as

$$\eta(x, t) = L_1 x^\alpha t^\beta, \qquad x \geqslant 0,\ t > 0, \tag{3.8}$$

$$u(x, t) = L t^\gamma F(\eta), \qquad t > 0, \tag{3.9}$$

where α, β, γ, L, L_1 are constants and the function $F(\eta)$ is differentiable up to order 2, which gives the restrictions on the parameters α, β, γ. These restrictions cannot be imposed precisely at this stage. The main problem now is to determine the similarity-characteristic relationship.

For this we will make use of the Theorem 3, characteristic equation (3.7) and relations (3.8) and (3.9). Since

$$\frac{dx}{dt} = M\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \tag{3.10}$$

thus, we can write from (3.9) and (3.8)

$$\begin{aligned}\frac{dx}{dt} &= M(x, t, F(\eta), F'(\eta))\\ &= M(t, \eta, F(\eta), F'(\eta)).\end{aligned} \tag{3.11}$$

Hence, taking into account the property that the constant of integration in (3.11) for the characteristic passing through $x=0$, $t=0$ equals zero,

$$x(t) = M_1(t, \eta, F(\eta), F'(\eta)), \tag{3.12}$$

where M_1 is an indefinite integral for M. On the basis of (3.3), (3.9), and Theorem 3, the relation (3.12) on the wave front assumes the form

$$x_w(t) = M_1(t, \eta_w, F'(\eta_w)). \tag{3.13}$$

On the other hand, from (3.8)

$$x_w(t) = \left(\frac{1}{L_1}\right)^{1/\alpha} t^{-(\beta/\alpha)} \eta_w^{1/\alpha}. \tag{3.14}$$

Combination of the equation (3.13) with (3.14) gives

$$\eta_w = M_2(L, L_1, \alpha, \beta, \gamma, \eta_w, F'(\eta_w)). \tag{3.15}$$

Thus, the location of η_w on the wavefront has been stated. We can easily observe that if the characteristic equation (3.7) does not depend on u, $\partial u/\partial x$, $\partial u/\partial t$, then M_1 depends on t and η only and η_w in (3.15) is given explicitly. However, in a general case η_w in (3.15) is given implicitly. Also, it may be noted that the relation (3.15) can be obtained only for partial differential equation of hyperbolic type and gives further restrictions on parameters involved in the governing equation. The illustration of the above procedure will be given by a couple of examples in the next part.

Now, we look for the similarity representation for partial boundary value problem, given by (3.1)–(3.6), under the similarity transformations (3.8) and (3.9). Making the use of these similarity transformations the derivatives of the function $u(x, t)$ can be expressed as

$$\frac{\partial u}{\partial x}(x, t) = \alpha L L_1^{1/\alpha} t^{\gamma+\beta/\alpha} \eta^{(\alpha-1)/\alpha} F'(\eta), \tag{3.16}$$

$$\frac{\partial^2 u}{\partial x^2}(x, t) = \alpha^2 L L_1^{2/\alpha} t^{\gamma+2\beta/\alpha} \eta^{(\alpha-1)/\alpha} \left[\frac{\alpha-1}{\alpha} \eta^{-1/\alpha} F'(\eta) + \eta^{(\alpha-1)/\alpha} F''(\eta)\right], \tag{3.17}$$

$$\frac{\partial u}{\partial t}(x, t) = L t^{\gamma-1}[\gamma F(\eta) + \beta \eta F'(\eta)], \tag{3.18}$$

$$\frac{\partial^2 u}{\partial t^2}(x, t) = L t^{\gamma-2}[\gamma(\gamma-1) F(\eta) + \beta(\beta+2\gamma-1) \eta F'(\eta) + \beta^2 \eta^2 F''(\eta)]. \tag{3.19}$$

Substitution (3.16)–(3.19) into the quasilinear hyperbolic partial differential equation (3.1) leads, in general case, to a nonlinear ordinary differential equation of second order with variable coefficients which can be expressed in the form

$$G(\alpha, \beta, \gamma, L, L_1, \eta, F(\eta), F'(\eta), F''(\eta)) = 0, \qquad 0 \leqslant \eta \leqslant \eta_w, \tag{3.20}$$

where η_w is given by relation (3.15). It may be noted also that in many cases the Eq. (3.20) becomes linear for which it is not difficult to find the exact solution. This is the fundamental advantage of similarity transformation.

To solve the ordinary boundary value problem we have to state the boundary conditions for it. The relation (3.8) implies that the point $x=0$ is transformed onto the point $\eta=0$ (the problem requires α, β, γ to be of suitable order and, indeed, they satisfy this requirement). Now, taking into account (3.18) and (3.9) the boundary conditions (3.2) and (3.2a) become, respectively,

$$F(0)=\frac{1}{\gamma}\,u_0(t)\,t^{1-\gamma}, \tag{3.21}$$

and

$$F(0)=u_0(t)\,t^{-\gamma}. \tag{3.22}$$

But $F(\eta=0)$ is a number, thus the function $u_0(t)$ has to have the following form:

$$u_0(t)=t^{\gamma-1}, \qquad t>0 \tag{3.23}$$

for boundary condition (3.2) and

$$u_0(t)=t^{\gamma}, \qquad t>0 \tag{3.24}$$

for boundary condition (3.2a).

The Theorem 3 and transformation (3.9) imply that boundary condition on the wavefront, (3.3), becomes

$$F(\eta=\eta_w)=0, \tag{3.25}$$

where η_w is given by (3.15). The initial conditions (3.5) and (3.6) are satisfied identically and do not give any more restrictions. Finally, the partial boundary value problems (3.1)–(3.6) is reduced to the ordinary boundary value problem,

$$G(\alpha,\beta,\gamma,L,L_1,\eta,F(\eta),F'(\eta),F''(\eta))=0, \qquad 0\leqslant\eta\leqslant\eta_w, \tag{3.26}$$

$$F(\eta=0)=\delta, \qquad \delta\neq 0, \tag{3.27}$$

$$F(\eta=\eta_w)=0, \tag{3.28}$$

where η_w is described by (3.15). Now we state the following theorem:

THEOREM 4. *If the function $F(\eta)$ is the solution of similarity representation of boundary value problem,*

$$G(\alpha,\beta,\gamma,L,L_1,\eta,F(\eta),F'(\eta),F''(\eta))=0, \qquad 0\leqslant\eta\leqslant\eta_w, \tag{3.29}$$

$$F(\eta=0)=\delta, \qquad \delta\neq 0, \tag{3.30}$$

$$F(\eta=\eta_w)=0, \tag{3.31}$$

then the function

$$u(x, t) = Lt^{\gamma} F(\eta) \tag{3.32}$$

is the solution of the hyperbolic partial boundary value problem

$$G_1\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\, \partial t}, \frac{\partial^2 u}{\partial t^2}\right) = 0, \tag{3.33}$$

$$\left\{\frac{\partial u}{\partial t}\,(x=0, t) \text{ or } u(x=0, t)\right\} = Lu_0(t), \qquad t > 0, \tag{3.34}$$

$$u(x = x_w(t), t) = 0, \qquad t > 0, \tag{3.35}$$

$$u(x, t=0) = 0, \qquad x > 0, \tag{3.36}$$

$$\frac{\partial u}{\partial t}\,(x, t=0) = 0, \qquad x > 0. \tag{3.37}$$

Where, in the above

$$\eta = L_1 x^{\alpha} t^{\beta}, \qquad x \geqslant 0,\ t > 0, \tag{3.38}$$

is similarity variable, η must satisfy similarity-characteristic relationship, α, β, γ, δ, L, L_1 are constants of similarity transformation and representation (with suitable restrictions), $u_0(t)$ satisfies the restriction (3.23) or (3.24) and further the differential form G, together with auxiliary conditions, is conformally invariant under r-parameter continuous group of transformation (2.5).

Proof. From the previous analysis it follows that the characteristic of quasilinear hyperbolic partial differential equation (3.33) passing through $x = 0$, $t = 0$ is described by (3.38) as

$$\eta = L_1 \frac{x^{\alpha}}{t^{m}}, \tag{3.39}$$

where $m = -\beta$ and η_w is given by (3.15). This curve gives the location of the wavefront. Thus, in the (x, t) space the domain of the partial boundary value problem (3.33)–(3.37) is the set A, Fig. 1. For further developments we will make use of a few lemmas which are stated and proved below.

LEMMA 1. *The set $\mathfrak{A}$ can be expressed in the form*

$$\mathfrak{A} = \bigcup_{\eta \in T} A_{\eta}, \tag{3.40a}$$

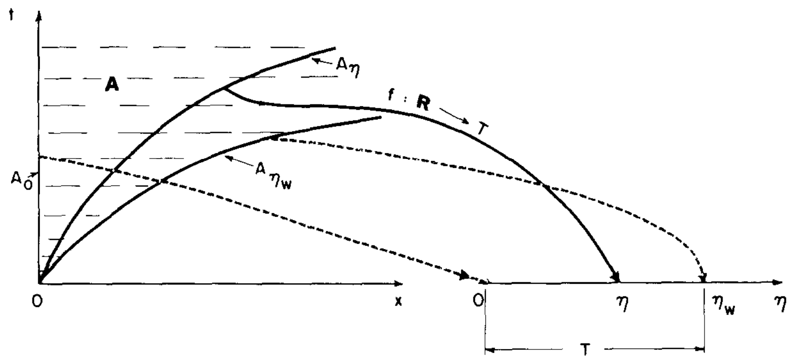

FIG. 1. The mapping $f: R \to T$ (x, t) space onto η space.

where

$$T = \langle 0, \eta_w \rangle, \tag{3.40b}$$

$$A_0 = \{(0, t) \colon t \geqslant 0\}, \tag{3.40c}$$

$$A_{\eta \neq 0} = \left\{(x, t) \colon L_1 \frac{x^\alpha}{t^m} = \eta,\ x > 0,\ t > 0\right\}, \tag{3.40d}$$

and $\bigcup_{\eta \in T} A_\eta$ *is the union of sets* A_η, $\eta \in T$ [13].

Proof. Let us take $(x_0, t_0) \in \mathfrak{A}$. If $(x_0, t) = (0, t_0)$, then $(x_0, t_0) \in A_0$, i.e., $(x_0, t_0) \in \bigcup_{\eta \in T} A_\eta$. Suppose that $(x_0, t_0) \not\subset A_0$, then from the definition of set $\mathfrak{A}$

$$0 < L_1 \frac{x_0^\alpha}{t_0^m} \leqslant \eta_w, \tag{3.40e}$$

and consequently $(x_0, t_0) \in A_{\eta 0}$, where

$$\eta_0 = L_1 \frac{x_0^\alpha}{t_0^m}. \tag{3.40f}$$

Hence,

$$\mathfrak{A} \subset \bigcup_{\eta \in T} A_\eta. \tag{3.40g}$$

On the other hand, for any $\eta \in T$ we have

$$0 \leqslant L_1 \frac{x^\alpha}{t^m} \leqslant \eta_w, \tag{3.40h}$$

which implies that

$$\forall(\eta \in T) \qquad (A_\eta \subset \mathfrak{A}), \tag{3.40i}$$

and consequently

$$\bigcup_{\eta \in T} A_\eta \subset \mathfrak{A}. \tag{3.40j}$$

The inclusions (3.40g) and (3.40j) give

$$\mathfrak{A} = \bigcup_{\eta \in T} A_\eta. \tag{3.40k}$$

Thus, Lemma 1 has been proved.

LEMMA 2. *For any two subsets A_{η_1}, $A_{\eta_2} \subset \mathfrak{A}$, whenever $\eta_1 \neq \eta_2$ then the intersection of A_{η_1} and A_{η_2} is empty.*

Proof. Suppose that η_1, $\eta_2 \in T$, $\eta_1 \neq \eta_2$ and $A_{\eta_1} \cap A_{\eta_2} \neq \varnothing$. Then, there exists $a_0 = (x_0, t_0)$ such that $a_0 \in A_{\eta_1}$ and $a_0 \in A_{\eta_2}$. But this implies that

$$L_1 \frac{x_0^\alpha}{t_0^m} = \eta_1 \qquad \text{and} \qquad L_1 \frac{x_0^\alpha}{t_0^m} = \eta_2, \tag{3.41a}$$

which gives

$$\eta_1 = \eta_2. \tag{3.41b}$$

The relation (3.41b) gives the contradiction to the above assumption. Lemma 2 has thus been proven.

Now we introduce the following:

DEFINITION. The subsets family $\mathfrak{R}$ generated by $\mathfrak{A}$ is the family of curves belonging to $\mathfrak{A}$ and crossing each characteristic exactly at one point, i.e.,

$$\mathfrak{R} = \{A_\eta : \eta \in T\} \tag{3.42}$$

(Fig. 1).

LEMMA 3. *The mapping $f: \mathfrak{R} \to T$, defined as*

$$f(A_\eta) = \eta, \tag{3.43a}$$

is one-to-one and onto.

Proof. The property onto of the mapping f follows directly from the definition of the set A_η (see Lemma 1) and the indexed family of sets $\mathfrak{R}$. Now, suppose that $A_{\eta_1} \neq A_{\eta_2}$. Then, following Lemma 2, $\eta_1 \neq \eta_2$, i.e.,

$$f(A_{\eta_1}) \neq f(A_{\eta_2}). \tag{3.43b}$$

Thus, the mapping f is one-to-one. Lemma 3 has been proven.

Lemma 3 states a very important property of the similarity transformation, namely, the similarity transformation maps boundary conditions (3.34) and (3.35) of the partial boundary value problem onto boundary conditions (3.30) and (3.31) of the ordinary boundary value problem, respectively.

Coming back to the proof of Theorem 4. The proof will be given by contradiction. Suppose that the function $u(x, t)$, (3.32), is not the solution of partial boundary value problem. Then, there are three possible cases:

(a) function $u(x, t) = Lt^\gamma F(\eta)$ does not satisfy the boundary condition (3.34), or

(b) function $u(x, t) = Lt^\gamma F(\eta)$ does not satisfy the boundary condition (3.35), or

(c) the function $u(x, t) = Lt^\gamma F(\eta)$ does not satisfy Eq. (3.33)

Case a. As indicated, in this case

$$\left\{u(x=0, t) \text{ or } \frac{\partial u}{\partial t}(x=0, t)\right\} \neq \{Lt^\gamma, Lt^{\gamma-1}\}. \tag{3.44}$$

On the basis of Lemma 3, (3.44) becomes

$$\{Lt^\gamma F(\eta=0), L\gamma t^{\gamma-1} F(\eta=0)\} \neq \{Lt^\gamma, Lt^{\gamma-1}\}, \tag{3.45}$$

or

$$\{F(\eta=0), F(\eta=0)\} \neq \{1, \delta\}. \tag{3.46}$$

Because the mapping f is one-to-one and onto it means that function $F(\eta)$ does not satisfy the boundary condition (3.30). Thus, function $F(\eta)$ is not the solution of the ordinary boundary value problem (3.29)–(3.31). This is the contradiction with the assumption of Theorem 4.

In a similar manner we obtain a contradiction with the assumption of Theorem 4 in the case (b). Case (c) is more complicated. Suppose the function $u(x, t)$ is not a solution of the partial differential equation (3.33). Thus, there exists a point $(x_0, t_0) \in A_{\eta_0} \subset \mathfrak{A}$ such that

$$G_1\left(x_0, t_0, u(x_0, t_0), \frac{\partial u}{\partial x}(x_0, t_0), \ldots, \frac{\partial^2 u}{\partial t^2}(x_0, t_0)\right) \neq 0. \tag{3.47}$$

Taking into account the similarity transformations and their properties (Lemma 1, 2), it can be easily verified that

$$G(\alpha, \beta, \gamma, L, L_1, \eta_0, F(\eta_0), F'(\eta_0), F''(\eta_0)) \neq 0, \tag{3.48}$$

where

$$\eta_0 = f(A_{\eta 0}) \qquad \text{and} \qquad \eta_0 \in T. \tag{3.49}$$

The inequality (3.48), together with (3.49), indicates that the function $F(\eta)$ does not satisfy the Eq. (3.29) over the interval $\langle 0, \eta_w \rangle$. This is the contradiction with the assumption of Theorem 4. Theorem 4 has thus been proven.

4. Analysis of Some Examples

PROBLEM 1. Let us consider the following hyperbolic quasilinear partial differential equation

$$\frac{E_0}{\rho q} x^n \left(-\frac{\partial u}{\partial x}\right)^{(1-q)/q} \frac{\partial^2 u}{\partial x^2} - \frac{E_0}{\rho} n x^{n-1} \left(-\frac{\partial u}{\partial x}\right)^{1/q} - \frac{\partial^2 u}{\partial t^2} = 0,$$

$$q > 0,\ x \geqslant 0,\ t \geqslant 0,\ E_0,\ \rho,\ q,\ n - \text{constants}, \tag{4.1a}$$

subjected to the auxiliary conditions

$$\frac{\partial u}{\partial t}(x = 0, t) = V_c t^{\delta_1}, \qquad t > 0,\ V_c,\ \delta_1\text{-constants}, \tag{4.1b}$$

$$u(x \geqslant x_w(t), t) = 0, \qquad t > 0, \tag{4.1c}$$

$$u(x, t = 0) = 0, \qquad x \geqslant 0, \tag{4.1d}$$

$$\frac{\partial u}{\partial t}(x, t = 0) = 0, \qquad x \geqslant 0, \tag{4.1e}$$

where $x_w(t)$ describes the wavefront. This system of equations arises for wave propagation in nonlinear-nonhomogeneous rods [14]. First, we look for multiparameter groups of transformations under which system (4.1) is invariant. To begin, consider a six-parameter continuous groups of transformations in the form

$$G_r\colon \begin{cases} S_r\colon \begin{cases} \bar{x} = A_x x, & \bar{t} = A_t t, \\ \bar{E}_0 = A_{E_0} E_0, & \overline{\rho q} = A_\rho \rho q, \qquad \bar{V}_c = A_{V_c} V_c, \end{cases} \\ \bar{u} = A_u u, \end{cases} \tag{4.2}$$

where A's are not to be regarded as independent, that is, S_r must constitute a subgroup of G_r which depends upon the same number of parameters as G_r. G_r may be enlarged via

$$(G_r^{E_2}-G_r):\begin{cases}\dfrac{\partial \bar{u}}{\partial \bar{x}}=A_u A_x^{-1}\dfrac{\partial u}{\partial x}, & \dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2}=A_u A_x^{-2}\dfrac{\partial^2 u}{\partial x^2},\\[2ex] \dfrac{\partial \bar{u}}{\partial \bar{t}}=A_u A_t^{-1}\dfrac{\partial u}{\partial t}, & \dfrac{\partial^2 u}{\partial \bar{t}^2}=A_u A_t^{-2}\dfrac{\partial^2 u}{\partial t^2}.\end{cases} \tag{4.3}$$

It turns out that the system (4.1) is conformally invariant under the group of transformation $G_r^{E_2}$, provided

$$\begin{aligned} A_u &= A_{E_0}^{q/(q-1)}A_\rho^{q/(1-q)}A_x^{(1+q-nq)/(1-q)}A_t^{2q/(q-1)},\\ A_{V_c} &= A_{E_0}^{q/(q-1)}A_\rho^{q/(1-q)}A_x^{(1+q-nq)/(1-q)}A_t^{(q+1)/(q-1)-\delta_1}.\end{aligned} \tag{4.4}$$

The substitution of (4.4) into (4.2) gives the following 4-parameter group of transformations under which the system (4.1) is conformally invariant:

$$G_4:\begin{cases} S_4:\begin{cases}\bar{x}=A_x x, \qquad \bar{t}=A_t t,\\ \bar{E}_0=A_{E_0}E_0, \qquad \overline{\rho q}=A_\rho \rho q,\\ \bar{V}_c=A_{E_0}^{q/(q-1)}A_\rho^{q/(1-q)}A_x^{(1+q-nq)/(1-q)}A_t^{(q+1)/(q-1)-\delta_1}V_c,\end{cases}\\ \bar{u}=A_{E_0}^{q/(q-1)}A_x^{(1+q-nq)/(1-q)}A_\rho^{q/(1-q)}A_t^{2q/(q-1)}u.\end{cases} \tag{4.5}$$

The matrix BC, (1.8a), has the form

$$BC:\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \dfrac{1+q-nq}{1-q}, & \dfrac{q+1}{q-1}-\delta_1, & \dfrac{q}{q-1}, & \dfrac{q}{1-q}\end{bmatrix} \tag{4.6}$$

with the rank $r=4$, i.e., the parameters A's in (4.5) are essential. It is also clear that the rank of the matrix C is $s=3<r$ and Theorem 2 indicates that the similarity transformation is available in the above boundary value problem (4.1); and because $p=s$ (see (1.5)) there is no absolute invariant determined solely from physical variables. Following Theorem 2, and taking into account matrix BC, (4.6), the absolute invariants for S_4 and G_4 are given by expressions of the form

$$G_4: \begin{cases} S_4: \left\{ \hat{\pi}_1 = K \dfrac{x^{(1+q-nq)/(1+q)}}{t^m}, \right. \quad K = \left(\dfrac{\rho q}{E_0}\right)^{q/(1+q)} \left(\dfrac{1}{V_c}\right)^{(1-q)/(1+q)}, \\ \qquad\qquad m = 1 + \delta_1 \dfrac{(1-q)}{1+q}, \qquad (4.7\text{a, b, c}) \\ \Pi_1 = u t^{-(\delta_1+1)} V_c^{-1}. \qquad (4.7\text{d}) \end{cases}$$

Hence, the similarity transformation of the problem (4.1) has the form

$$u(x,t) = V_c t^{\delta_1+1} F(\eta), \tag{4.8a}$$

$$\eta = K \frac{x^{(1+q-nq)/(1+q)}}{t^m}, \; 1+q-nq>0, \; q>0, \; m>0, \qquad (4.8\text{b, c, d, e})$$

where $\eta \equiv \hat{\pi}_1$, $F(\eta) \equiv \Pi_1$ and K and m are constants given by (4.7b, c).

Making use of (3.16), (3.18), (3.13), and (4.8), after some algebra, the similarity-characteristic relationship (3.15) takes the form

$$\eta_w = \left(\frac{1+q-nq}{1+q}\right)^{(1+q)/2q} \frac{1}{m} [-F'(\eta_w)]^{(1-q)/2q}, \tag{4.9}$$

where

$$q>0, \quad m>0, \quad n<\frac{1+q}{q}. \tag{4.10}$$

From (4.9) we see that in the problem of wave propagation in a nonlinear-nonhomogeneous rod subjected to time-dependent velocity impact the similarity-characteristic relationship is given implicitly, however, for linear-nonhomogeneous case wherein $\eta_w = (2-n)/2$, it is given explicitly. The similarity representation (3.20) becomes

$$\begin{aligned} &\left[\left(\frac{1+q-nq}{1+q}\right)^{(1+q)/q} (-F'(\eta))^{(1-q)/q} - m^2\eta^2\right] F''(\eta) \\ &\quad - \left(\frac{1+q-nq}{1+q}\right)^{1/q} \frac{nq^2}{1+q} \eta^{-1} [-F'(\eta)]^{1/q} - m(m-2\delta_1-1)\,\eta \\ &\quad \times F'(\eta) - \delta_1(\delta_1+1)\,F(\eta) = 0, \qquad 0 \leqslant \eta \leqslant \eta_w, \end{aligned} \tag{4.11a}$$

where η_w is given by (4.9). The boundary condition (3.21), on the basis of (4.1b) assumes the form

$$F(\eta=0) = \frac{1}{1+\delta_1}, \tag{4.11b}$$

and the condition on the wave front is

$$F(\eta=\eta_w)=0. \tag{4.11c}$$

Similarity representation (4.9)–(4.11) is a nonlinear boundary value problem, which can be solved numerically for the values of q away from unity. However, for simpler cases: (a) linear nonhomogeneous case with time-dependent velocity impact ($q=1$), (b) linear homogeneous case ($q=1$, $n=0$), and (c) nonlinear homogeneous case with constant velocity impact ($n=0$, $\delta=0$), the solutions of the system (4.9)–(4.11) are available in an exact form. For instance, the solutions in cases (a) and (b) are given in [14] and have the form:

Case a (Linear Nonhomogeneous Case).

$$F(\eta)=\frac{1}{1+\delta_1}\left[F_1(\eta)-\frac{F_1(\eta_w)}{F_2(\eta_w)}F_2(\eta)\right], \qquad 0\leqslant\eta\leqslant\eta_w=\frac{2-n}{2} \tag{4.12}$$

and under the condition that the parameters δ and n must satisfy the inequalities

$$(n-4)-2(2-n)\,\delta_1<0 \qquad \text{and} \qquad n<1, \tag{4.13}$$

where the functions $F_1(\eta)$ and $F_2(\eta)$ assume the form

$$F_1(\eta)=1+\sum_{s=1}^{\infty}\eta^{2s}\frac{1}{(2-n)^s\,s!} \times\frac{(\delta_1+1)\,\delta_1(\delta_1-1)(\delta_1-2)\cdots[\delta_1-(2s-3)][\delta_1-2(s-1)]}{(3-n)(5-2n)\cdots[(2-n)\,s+n-1]}, \tag{4.14a}$$

$$\begin{aligned}F_2(\eta)=\eta^{(2-2n)/(2-n)}\Bigg\{1+\sum_{s=1}^{\infty}\eta^{2s}\frac{1}{(2-n)^s\,s!}&\\ \times\left(\left(\delta_1+\frac{n}{2-n}\right)\left(\delta_1-1+\frac{n}{2-n}\right)\Big/(3-2n)\right)&\\ \times\cdots\times\left[\delta_1-2(s-1)+\frac{n}{2-n}\right]&\\ \times\left[\delta_1-(2s-1)+\frac{n}{2-n}\right]\Big/[(2-n)\,s+1-n]\Bigg\},&\end{aligned} \tag{4.14b}$$

Case b (Linear Homogeneous Case).

$$F(\eta)=\frac{1}{1+\delta_1}(1-\eta)^{1+\delta_1}, \qquad 0\leqslant\eta\leqslant 1,\ \delta_1>-1. \tag{4.14c}$$

On the basis of Theorem 4 the function $u(x,t)=V_c t^{\delta_1+1}F(\eta)$ is the solution of partial boundary value problem (4.1). Indeed, it is not difficult to check that the function $u(x,t)$ satisfies partial differential equation (4.1a) and auxiliary conditions (4.1b)–(4.1e) [14].

Case c (Nonlinear Homogeneous Case). Nonlinear homogeneous case was considered in [9].

The solution for similarity representation has a form

$$F(\eta)=1-\eta, \qquad 0\leqslant\eta\leqslant 1,\ q>0, \tag{4.15}$$

and after inverse transformation it can be easily verified that $u(x,t)$ satisfies boundary value problem (4.1).

PROBLEM 2. Let us consider linear partial differential equation

$$x^n\frac{\partial^2\sigma}{\partial x^2}=\frac{\partial^2\sigma}{\partial\tau_2}, \qquad x\geqslant 0,\ \tau\geqslant 0,\ n\text{-constant}, \tag{4.16a}$$

subjected to the auxiliary conditions

$$\sigma(x=0,\tau)=\sigma_c\tau^{\delta_2}, \qquad \tau>0,\ \sigma_c;\delta_2-\text{constants}, \tag{4.16b}$$

$$\sigma(x\geqslant x_w(t),\tau)=0, \qquad \tau>0, \tag{4.16c}$$

$$\sigma(x,\tau=0)=0, \qquad x\geqslant 0, \tag{4.16d}$$

$$\frac{\partial\sigma}{\partial\tau}(x,\tau=0)=0, \qquad x\geqslant 0, \tag{4.16e}$$

where $x_w(t)$ describes the wavefront, $\tau=c_0t$, c_0 – constant. The above system of equations arises for wave propagation in linear non-homogeneuous rods subjected to time-dependent stress impact [15].

It can be easily shown that starting from G_r, a four-parameter continuous groups of transformations

$$G_r\colon\begin{cases} S_r\colon\begin{cases}\bar{x}=A_x x, \qquad \bar{\tau}=A_\tau\tau,\\ \bar{\sigma}_c=A_{\sigma_c}\sigma_c,\end{cases}\\ \bar{\sigma}=A_\sigma\sigma,\end{cases} \tag{4.17}$$

the system of equations (4.16) is conformally invariant under $G_r^{E_2}$, an enlargement of (4.17), provided

$$A_\tau = [A_x]^{(2-n)/2}, \tag{4.18a}$$

$$A_\sigma = A_{\sigma_c}[A_x]^{\delta_2(2-n)/2}. \tag{4.18b}$$

The substitution (4.18) into (4.17) gives the following 2-parameter groups of transformations under which the system (4.16) is conformally invariant:

$$G_2: \begin{cases} S_2: \begin{cases} \bar{x} = A_x x, & \bar{\tau} = [A_x]^{(2-n)/2}\tau, \\ \bar{\sigma}_c = A_{\sigma_c}\sigma_c, \end{cases} \\ \bar{\sigma} = [A_x]^{\delta_2(2-n)/2} A_{\sigma_c}\sigma. \end{cases} \tag{4.19}$$

The matrix BC has the form

$$\begin{bmatrix} 1 & 0 \\ \dfrac{2-n}{2} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.20}$$

With the rank $r=2$, i.e., the parameters A's are essential. The rank of the matrix C is $s=1<r$, i.e., the similarity transformation can be determined for the boundary value problem (4.16); and because $p=s$, there is no absolute invariant determined solely from physical variables. Following Theorem 2, and taking into account matrix BC, (4.20), the absolute invariants for S_2 and G_2 are given by expressions of the form

$$G_2: \begin{cases} S_2: \left\{ \hat{\pi}_1 = \dfrac{x^{(2-n)/2}}{\tau}, \right. & (4.21a) \\ \Pi_1 = \sigma\tau^{-\delta_2}\sigma_c^{-1}. & (4.21b) \end{cases}$$

Hence, the similarity transformation for the system (4.16) assumes the form

$$\sigma(x, \tau) = \sigma_c \tau^{\delta_2} f(\eta), \tag{4.22a}$$

$$\eta = \frac{x^{(2-n)/2}}{\tau}, \qquad n<2, \tag{4.22b}$$

where $\eta \equiv \hat{\pi}_1$, $f(\eta) \equiv \Pi_1$.

Making the use of (3.16), (3.18), (3.13), and (4.22), the similarity-characteristic relationship (3.15) assumes the form

$$\eta_w = \frac{2-n}{2}, \qquad n<2, \tag{4.23}$$

i.e., η_w is given explicitly. The similarity representation (3.20) on the basis of the equation of motion (4.16a) becomes

$$\eta[(2-n)^2-4\eta^2]\,f''(\eta)+[8\eta^2(\delta_2-1)-n(2-n)]\,f'(\eta)$$
$$-4\delta_2(\delta_2-1)\,\eta f(\eta)=0, \qquad 0\leqslant\eta\leqslant\eta_w. \tag{4.24a}$$

The boundary condition (4.16b) on the basis of (3.22) has the form

$$f(\eta=0)=1, \tag{4.24b}$$

and the condition on the wavefront becomes

$$f(\eta=\eta_w)=0, \qquad \eta_w=\frac{2-n}{2},\ n<2. \tag{4.24c}$$

The solution of the similarity representation (4.24a), (4.24b), (4.24c) is given in [15] and has the form

$$f(\eta)=f_1(\eta)-\frac{f_1(\eta_w)}{f_2(\eta_w)}\,f_2(\eta), \qquad 0\leqslant\eta\leqslant\eta_w=\frac{2-n}{2}. \tag{4.25}$$

The parameters n and δ_2 have to satisfy the conditions

$$n+2n\delta_2-4\delta_2<0, \qquad n<2,\ \delta_2 \text{ is a nonnegative integer}, \tag{4.26a}$$

and

$$n+2n\delta_2-4\delta_2<0, \qquad n<2,\ n\neq 2-\frac{1}{s},\ \delta_2 \text{ is not}, \tag{4.26b}$$

a nonnegative integer and s is a positive integer. The functions $f_1(\eta)$ and $f_2(\eta)$ in (4.25) assume the form

$$f_1(\eta)=1+\sum_{s=1}^{\infty}\eta^{2s}\,\frac{\delta_2(\delta_2-1)(\delta_2-2)\cdots(\delta_2-2(s-1)(\delta_2-(2s-1))}{s!(2-n)^s\,(1-n)(3-2n)\cdots[(2-n)\,s-1]}, \tag{4.27}$$

$$f_2(\eta)=\eta^{2/(2-n)}\Bigg\{1+\sum_{s=1}^{\infty}\eta^{2s}\,\frac{1}{(2-n)^s\,s!}$$
$$\times\left(\left(\delta_2-\frac{2}{2-n}\right)\left(\delta_2-1-\frac{2}{2-n}\right)\Big/(3-n)\right)$$
$$\times\left(\delta_2-2-\frac{2}{2-n}\right)\left(\delta_2-3-\frac{2}{2-n}\right)\Big/(5-2n)$$
$$\times\cdots\times\left(\delta_2-2(s-1)-\frac{2}{2-n}\right)$$
$$\times\left(\delta_2-(2s-1)-\frac{2}{2-n}\right)\Big/[(2-n)\,s+1]\Bigg\}. \tag{4.28}$$

On the basis of Theorem 4 the function $\sigma(x, \tau) = \sigma_c \tau^{\delta_2} f(\eta)$ is the solution of the partial bounary value problem (4.16), which can be easily verified [15].

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